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1978 J. Phys. A: Math. Gen. 11 1715

(http://iopscience.iop.org/0305-4470/11/9/006)

J. Phys. A: Math. Gen., Vol. 11, No. 9, 1978. Printed in Great Britain.

Multiplicity distributions of created bosons: the method of combinants†

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Received 3 April 1978

Abstract. The number of bosons that are created by an excitation process, in a system which initially has none, is a random variable having a discrete probability distribution, defined on the non-negative integers, which satisfies $P(0)>0$. The logarithm of the resulting probability generating function is therefore analytic at the origin, and the series expansion coefficients thereby generated can each be expressed as a finite combination of ratios of the $P(n)$, giving them an interestingly close kinship to experimental data. These 'combinants' are additive for sums of boson multiplicity random variables which are independent, and they all vanish except for the first-order one (which has the value of the mean) in the important Poisson case. The combinants are readily calculated in a number of theoretical models for created boson multiplicities, including the thermal model, some of the related chaotic radiation models, and in some models described by master rate equations. The resulting boson multiplicity distributions are then described by the very general convoluted multiple Poisson distribution, which is expressed directly in terms of the combinants. Combinants thus provide a neat theoretical tool for dealing with many models of boson multiplicity distributions, some of which had previously seemed intractable. It is further shown that cumulants have the same formal relation to combinants as moments have to probabilities. Combinants are a probabilistic tool which seems uniquely well suited to both the experimental and theoretical study of multiplicity distributions of created bosons.

1. Introduction and basic development

Collisions or other excitation processes often create bosons in systems where *none were present initially*. We call $P(n)$, the probability that n bosons were produced by the process, the *created boson multiplicity distribution*. The domain of $P(n)$ is the non-negative integers ($n=0, 1, 2, \ldots$), and it seems reasonable to postulate, bearing in mind any reasonable quantum description of boson creation (Bjorken and Drell 1965), that there is a non-zero probability to produce no bosons at all, i.e.

$$P(0)>0. \tag{1.1}$$

We have, in addition, the usual probability distribution conditions

$$P(n)\geqslant 0, \qquad n=1,2,\ldots \tag{1.2a}$$

and

$$\sum_{n=0}^{\infty} P(n)=1. \tag{1.2b}$$

† This work was supported by Nuclear Physics, Nuclear Science Division, of the US Department of Energy.

A very useful tool for studying $P(n)$ is the generating function

$$F(\lambda) \equiv \sum_{n=0}^{\infty} \lambda^n P(n). \tag{1.3}$$

Simple models of boson production often predict the Poisson multiplicity distribution (Bjorken and Drell 1965)

$$P(n) = \frac{(\bar{n})^n}{n!} e^{-\bar{n}} \tag{1.4}$$

which has the simple generating function

$$F(\lambda) = \exp[(\lambda - 1)\bar{n}]. \tag{1.5}$$

Indeed, it is useful to try to characterise the general $P(n)$ in terms of its deviations from the Poisson. Traditionally, this is done by considering deviations which the *moments* of $P(n)$ have from special properties possessed by Poisson moments (Burington and May 1953). On a different tack, we note that the natural logarithm of the Poisson generating function, equation (1.5), is a simple first degree polynomial. Non-zero second- and higher-order coefficients of the power series expansion, in λ, of the natural logarithm of the generating function, spell out the deviations of a general created boson multiplicity distribution from the Poisson. We write this expansion as follows

$$\ln(F(\lambda)) = \ln(F(0)) + \sum_{k=1}^{\infty} C(k)\lambda^k \tag{1.6a}$$

$$= \ln(P(0)) + \sum_{k=1}^{\infty} C(k)\lambda^k. \tag{1.6b}$$

It is seen that our condition (1.1) is necessary for this expansion to exist. We note, from equations (1.2*b*) and (1.3) that $F(1) = 1$. Thus we arrive at

$$\ln(P(0)) = -\sum_{k=1}^{\infty} C(k) \tag{1.7}$$

which implies that

$$\ln(F(\lambda)) = \sum_{k=1}^{\infty} C(k)(\lambda^k - 1) \tag{1.8a}$$

or

$$F(\lambda) = \exp\left(\sum_{k=1}^{\infty} C(k)(\lambda^k - 1)\right). \tag{1.8b}$$

The expansion coefficients $C(1)$, $C(2)$, ... thus completely characterise $P(n)$. For a Poisson, $C(1)$ is the mean, while $C(2)$, $C(3)$, ... all vanish. Let us try to calculate $C(1)$, $C(2)$, ... in the general case. We proceed by the method of inserting one power series into another

$$\ln(F(\lambda)) = \ln(P(0)) + \ln\left(1 + \sum_{n=1}^{\infty} \lambda^n \frac{P(n)}{P(0)}\right) \tag{1.9a}$$

$$= \ln(P(0)) + \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j}\left(\sum_{n=1}^{\infty} \lambda^n \frac{P(n)}{P(0)}\right)^j. \tag{1.9b}$$

From equations (1.6*b*) and (1.9*b*) we may read off

$$C(1)=\left(\frac{P(1)}{P(0)}\right) \tag{1.10a}$$

$$C(2)=\left(\frac{P(2)}{P(0)}\right)-\frac{1}{2}\left(\frac{P(1)}{P(0)}\right)^2 \tag{1.10b}$$

$$C(3)=\left(\frac{P(3)}{P(0)}\right)-\left(\frac{P(1)}{P(0)}\right)\left(\frac{P(2)}{P(0)}\right)+\frac{1}{3}\left(\frac{P(1)}{P(0)}\right)^3 \tag{1.10c}$$

etc.

We note that each $C(k)$ is expressible in terms of just the *first k* probability *ratios* $(P(1)/P(0))$, $(P(2)/P(0)), \ldots, (P(k)/P(0))$. This stands in stark contrast to 'ordinary' probability coefficients such as moments and cumulants, each of which involves *every single one* of the *infinite* number of $P(n)$ in its definition. However, note again the *necessity* of condition (1.1) to the existence of the $C(k)$.

Among the traditional probability coefficients, the cumulants, the first two of which are the mean and the variance, occupy a special role. This is partly because any random variable which is composed of a sum of independent random variables, has for its cumulants just the sum of the respective cumulants of its components (Burington and May 1953). This coefficient 'additivity property' is, however, shared by the $C(k)$.

Given random variables $N_1, N_2, \ldots$, where N_i represents the number of bosons of the ith type (e.g. having momentum $\boldsymbol{p}_i$) which are independently created by a certain process, we wish to consider the distribution of N, a sum random variable, representing the total number of bosons of a certain class which are created,

$$N=\sum_i N_i. \tag{1.11}$$

Given that each N_i is independently distributed according to $P_i(n)$, which has generating function $F_i(\lambda)$, it is easily shown that N is distributed according to $P(n)$, whose generating function $F(\lambda)$ is just the product of the $F_i(\lambda)$,

$$F(\lambda)=\prod_i F_i(\lambda). \tag{1.12}$$

Thus, if we write each generating function $F_i(\lambda)$ in terms of its coefficients $C_i(k)$, $k=1,2,\ldots$, as in equation (1.8*b*), we see from equation (1.12) that the coefficients $C(k)$ of N satisfy the 'additivity property'

$$C(k)=\sum_i C_i(k), \qquad k=1,2,\ldots. \tag{1.13}$$

The $C(k)$ share this important property with the cumulants, but cumulants are finitely expressible in terms of the *moments*, each of which involves an infinite 'cumulative' sum over all the $P(n)$. The $C(k)$, however, are expressible directly as a *finite combination* of *ratios* of the $P(n)$. In view of the similarities and distinctions, we feel it is appropriate to call the $C(k)$ 'combinants'.

For the study of created boson multiplicity distribution in particular, combinants appear to be a more appropriate tool than the traditional (and more widely applicable) moment based probability coefficients. This is because : (1) they trivially exhibit the deviations from the Poisson; (2) they possess the elegant and very useful additivity property; and (3) the first k combinants follow immediately from the *first* $k+1$ *unnormalised probabilities*—precisely the nature of created boson multiplicity data provided by experiment.

For practical applications, we need the general formula for the $C(k)$, whose derivation was begun in equation (1.9). Continuing with this matter here, we apply the multinomial expansion to the jth power of the innermost sum in equation (1.9b), and then re-arrange the result in ascending powers of λ:

$$\sum_{k=1}^{\infty} C(k)\lambda^k = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j}\left(\sum_{n=1}^{\infty} \frac{P(n)}{P(0)}\lambda^n\right)^j \tag{1.14a}$$

$$= \sum_{j=1}^{\infty} (-1)^{j+1}(j-1)! \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \left[\prod_{q=1}^{\infty} \frac{(\lambda^q)^{n_q}}{n_q!}\left(\frac{P(q)}{P(0)}\right)^{n_q}\right]\delta\left(j, \sum_{r=1}^{\infty} n_r\right) \tag{1.14b}$$

$$= -\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \left(-1+\sum_{r=1}^{\infty} n_r\right)!\left[\prod_{q=1}^{\infty} \frac{(\lambda^q)^{n_q}}{n_q!}\left(-\frac{P(q)}{P(0)}\right)^{n_q}\right]\theta\left(-1+\sum_{r=1}^{\infty} n_r\right) \tag{1.14c}$$

$$= \sum_{k=1}^{\infty} \lambda^k\left\{-\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \left(-1+\sum_{r=1}^{\infty} n_r\right)!\left[\prod_{q=1}^{\infty} \frac{1}{n_q!}\left(-\frac{P(q)}{P(0)}\right)^{n_q}\right]\delta\left(k, \sum_{r=1}^{\infty} rn_r\right)\right\}. \tag{1.14d}$$

Here δ is the Kronecker delta and θ is the Heaviside unit step function, but specified to have value unity at zero. The general formula for $C(k)$, then, is

$$C(k) = -\sum_{n_1=0}^{k} \cdots \sum_{n_p=0}^{[k/p]} \cdots \sum_{n_k=0}^{1} \left(-1+\sum_{r=1}^{k} n_r\right)!\left[\prod_{q=1}^{k} \frac{1}{n_q!}\left(-\frac{P(q)}{P(0)}\right)^{n_q}\right]\delta\left(k, \sum_{r=1}^{k} rn_r\right),$$
$$k = 1, 2, \ldots, \tag{1.15}$$

where $[k/p]$ stands for the integer part of the quotient (k/p).

In order to program equation (1.15) on a computer, one needs an algorithm which sequentially generates the full set of non-negative integer k-tuples $(n_1, n_2, \ldots, n_k)$ which satisfy

$$\sum_{r=1}^{k} rn_r = k. \tag{1.16}$$

A fairly simple such algorithm is described in the appendix. We emphasise again that computation of the kth combinant *only* requires knowledge of all the *unnormalised probabilities* for having produced *q bosons, where* $q \leqslant k$. We thus believe that combinant analysis will be a worthwhile addition to the methods of experimental data analysis.

In the next section we shall see that certain theoretical models for created boson multiplicity distributions are most naturally expressed in terms of the combinants—due in considerable part to the additivity property.

2. The convoluted multiple Poisson and some theoretical applications

In this section we shall see that a number of theoretical models for created boson multiplicity distributions yield relatively simple expressions for the combinants. It is of interest, then, to have a general formula for the $P(n)$ in terms of the $C(k)$. Equation (1.8b) gives $F(\lambda)$ as an exponentiated series in terms of the $C(k)$. We may apply our technique of inserting one series into another, re-expressing the innermost series to a power by using the multinomial expansion, and re-ordering the result in ascending powers of λ, to find the $P(n)$. First we note that since from equation (1.7)

$$P(0)=\exp\left(-\sum_{k=1}^{\infty} C(k)\right), \tag{2.1}$$

it follows, from equations (1.3), (1.8*b*), and the techniques illustrated in equation (1.14), that

$$\sum_{n=0}^{\infty} \lambda^n \frac{P(n)}{P(0)} = \exp\left(\sum_{k=1}^{\infty} C(k)\lambda^k\right) \tag{2.2a}$$

$$= \sum_{j=0}^{\infty} \frac{1}{j!}\left(\sum_{k=1}^{\infty} C(k)\lambda^k\right)^j \tag{2.2b}$$

$$= \sum_{j=0}^{\infty} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \left[\prod_{k=1}^{\infty} \frac{(C(k)\lambda^k)^{n_k}}{n_k!}\right] \delta\left(j, \sum_{r=1}^{\infty} n_r\right) \tag{2.2c}$$

$$= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \left(\prod_{k=1}^{\infty} \frac{(\lambda^k)^{n_k}(C(k))^{n_k}}{n_k!}\right) \tag{2.2d}$$

$$= \sum_{n=0}^{\infty} \lambda^n \left[\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \left(\prod_{k=1}^{\infty} \frac{(C(k))^{n_k}}{n_k!}\right) \delta\left(n, \sum_{r=1}^{\infty} r n_r\right)\right]. \tag{2.2e}$$

Thus, from equation (2.2), for $n = 1, 2, 3, \ldots$,

$$P(n) = P(0) \sum_{n_1=0}^{n} \cdots \sum_{n_p=0}^{[n/p]} \cdots \sum_{n_n=0}^{1} \left(\prod_{k=1}^{n} \frac{(C(k))^{n_k}}{n_k!}\right) \delta\left(n, \sum_{r=1}^{n} r n_r\right) \tag{2.3}$$

while $P(0)$ is given by equation (2.1). Thus, we have expressed an arbitrary created boson multiplicity distribution directly in terms of its combinants.

A very suggestive way of writing this result for $P(n)$ is

$$P(n) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \left(\prod_{k=1}^{\infty} \frac{(C(k))^{n_k}\, \mathrm{e}^{-C(k)}}{n_k!}\right) \delta\left(n, \sum_{r=1}^{\infty} r n_r\right), \tag{2.4}$$

which is a convoluted form of the multiple Poisson distribution (Gyulassy and Kauffmann 1978) with the combinants, $C(k)$, as the 'means'. This is certainly a most intriguing way of viewing equation (2.4), which is, after all, the most general possible form of the created boson multiplicity distribution. A bit of caution regarding such a viewpoint is in order, however. First, unlike true means of a multiple Poisson, some $C(k)$ are permitted to assume negative values, at least for $k \geq 2$ (see equations (1.10) and (1.15)). Second, the nature of the convolution is rather unorthodox—an orthodox convolution would make $P(n)$ itself a Poisson, which is definitely not the case unless all the $C(k)$ for $k \geq 2$ vanish!

With the very general convoluted multiple Poisson distribution in mind, let us consider some theoretical models of created boson multiplicity distributions. If the bosons are created by the action of a classical source current, it is well known that they will be independently Poisson distributed (only $C(1) = \bar{n}$ non-vanishing) in each mode (Bjorken and Drell 1965). It is the simplest consequence of the additivity property that the multiplicity distribution of such bosons over any range of modes will also be Poisson distributed, with the mean number being the sum of the means in each mode.

A common theme in many theoretical models is that the bosons are created with independent distributions in each mode. The multiplicity distribution over a range of modes follows, of course, from its combinants, and these, by additivity, are just the sums of the respective combinants from each mode. For example, a large random ensemble of classical sources sufficiently dispersed over space and time tends, in the sense of a limiting *average* over the boson Poisson distributions produced by each source ensemble member, to create bosons having independent geometric (Bose–Einstein) distributions in each mode (Klauder and Sudarshan 1968). Suppose the ith mode is Bose–Einstein distributed with mean $\bar{n}_i$. Then

$$P_i(n)=\frac{(\bar{n}_i)^n}{(1+\bar{n}_i)^{n+1}}, \tag{2.5}$$

which has the generating function

$$F_i(\lambda)=\left(\frac{1-[\bar{n}_i/(1+\bar{n}_i)]}{1-\lambda[\bar{n}_i/(1+\bar{n}_i)]}\right) \tag{2.6a}$$

$$=\exp\left\{\ln\left[1-\left(\frac{\bar{n}_i}{1+\bar{n}_i}\right)\right]-\ln\left[1-\lambda\left(\frac{\bar{n}_i}{1+\bar{n}_i}\right)\right]\right\} \tag{2.6b}$$

$$=\exp\left[\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{\bar{n}_i}{1+\bar{n}_i}\right)^k(\lambda^k-1)\right]. \tag{2.6c}$$

Thus the combinants for the ith mode (Bose–Einstein) distribution are

$$C_i(k)=\frac{1}{k}\left(\frac{\bar{n}_i}{1+\bar{n}_i}\right)^k, \qquad k=1,2,\ldots. \tag{2.7}$$

Therefore the multiplicity distribution over a range of modes is just the *convoluted multiple Poisson* (equations (2.1), (2.3), and (2.4)) whose combinants are

$$C(k)=\frac{1}{k}\sum_i\left(\frac{\bar{n}_i}{1+\bar{n}_i}\right)^k, \qquad k=1,2,\ldots. \tag{2.8}$$

To illustrate this result, consider the special example of identical scalar bosons created in thermal equilibrium at temperature T within a box of volume V. Statistical mechanics yields independent Bose–Einstein distributions of such bosons in each mode, with means (Klauder and Sudarshan 1968)

$$\bar{n}_i=\{\exp[(|\boldsymbol{p}_i|^2c^2+\mu^2c^4)^{1/2}/k_BT]-1\}^{-1} \tag{2.9}$$

where k_B is the Boltzmann constant and μ is the boson mass. The *total* number of bosons created thus within the box is distributed according to a *convoluted multiple Poisson* whose combinants, according to equation (2.8) and (2.9), are

$$C(k)=\frac{1}{k}\sum_i\exp[-k(|\boldsymbol{p}_i|^2c^2+\mu^2c^4)^{1/2}/k_BT] \tag{2.10a}$$

$$=\frac{1}{k}\frac{V}{(2\pi\hbar)^3}\int d^3\boldsymbol{p}\,\exp[-k(|\boldsymbol{p}|^2c^2+\mu^2c^4)^{1/2}/k_BT] \tag{2.10b}$$

$$=\frac{1}{k}\left(\frac{V\mu^3c^3}{2\pi^2\hbar^3}\right)\left(\frac{k_BT}{k\mu c^2}\right)K_2\left(\frac{k\mu c^2}{k_BT}\right), \qquad k=1,2,\ldots, \tag{2.10c}$$

where K_2 is the modified Bessel function (Gyulassy and Kauffmann 1978). It is to be noted that in the absence of the combinant and convoluted multiple Poisson concepts, the thermal boson total multiplicity distribution seems virtually intractable.

Within the context of these 'independent Bose–Einstein per mode' models, often called chaotic radiation models, it is interesting to remark on the physical interpretation of the combinants. First, the quantity $kC_i(k)$ is, from equations (2.7) and (2.5), just the probability that the occupation number of the ith mode is greater than or equal to k. Thus, the quantity $kC(k)$ is, from equation (2.8), the mean number of modes, in the range being considered, which have occupation number greater than or equal to k (Gyulassy and Kauffmann 1978). In the thermal model, one would, then, expect $C(k)$ to fall exponentially with k for large k, which is indeed the case, as can be seen from equation (2.10*c*). (Note, however, than an exponential fall-off of the combinants does *not* occur in such a thermal model for the *total* number of bosons if they are *massless*, because massless bosons can populate a great many sufficiently low-lying modes in *any* numbers at little cost in energy.) It is worthwhile to point out, with respect to the class of 'independent Bose–Einstein per mode' models, that if every mode in the range under consideration is sparsely populated (all $\bar{n}_i < 1$), we can reasonably expect $C(k) \ll C(1)$ for all $k \geq 2$, and our *convoluted multiple Poisson* distribution over the range of modes in question to be fairly well approximated by an *ordinary* Poisson (Hagedorn 1973).

Let us now turn our attention to quite a different class of models for created boson multiplicity distributions. Instead of postulating the distributions in each mode on the basis of approximate quantum and statistical notions, one simply assumes that there exist time-dependent *total* rates (over the whole desired range of modes) for certain types of *instantaneous* boson emission and absorption, which can be manipulated according to *classical* probabilistic precepts (Malfliet and Karant 1975, Gyulassy and Kauffmann 1978). This sort of crude procedure typically results in probability *transport* or *master rate* equations. Here we shall assume that $\gamma_k^+(t)$ is the time-dependent rate for the instantaneous emission of a group of k bosons (e.g. from the decay of a k-boson resonance), while $\gamma^-(t)$ is the time-dependent rate for instantaneous single boson absorption. Our created boson multiplicity distribution is assumed to vary over an infinitesimal time according to the dictates of *classical* probability notions (Gyulassy and Kauffmann 1978):

$$P(n, t+\mathrm{d}t) = (n+1)\gamma^-(t)\,\mathrm{d}t\,P(n+1, t) + \sum_{k=1}^{\infty} \gamma_k^+(t)\,\mathrm{d}t\,P(n-k, t) + P(n, t)\left(1 - n\gamma^-(t)\,\mathrm{d}t - \sum_{k=1}^{\infty} \gamma_k^+(t)\,\mathrm{d}t\right) \tag{2.11a}$$

or

$$\frac{\mathrm{d}P(n, t)}{\mathrm{d}t} = \gamma^-(t)[(n+1)P(n+1, t) - nP(n, t)] + \sum_{k=1}^{\infty} \gamma_k^+(t)(P(n-k, t) - P(n, t)). \tag{2.11b}$$

It is understood in equation (2.11) that $P(n, t)$ vanishes identically for $n < 0$. We have the normalisation condition

$$\sum_{n=0}^{\infty} P(n, t) = 1 \tag{2.12a}$$

and the condition that no bosons are initially present (we consider only created boson multiplicity distributions)

$$P(n, t=0) = \delta_{n0}. \tag{2.12b}$$

The master rate equation (2.11) does have the advantage over previously discussed models that it makes an attempt to include the effect of *boson emission in dynamically correlated groups*. We shall see that this phenomenon alone forces the presence of higher combinants in the solution.

We introduce the time-dependent generating function

$$F(\lambda, t) \equiv \sum_{n=0}^{\infty} \lambda^n P(n, t) \tag{2.13}$$

which satisfies the boundary condition $F(\lambda=1, t)=1$ from equation (2.12*a*) and the initial condition $F(\lambda, t=0)=1$ from equation (2.12*b*). In terms of $F(\lambda, t)$, equation (2.11*b*) becomes

$$\frac{\partial F(\lambda, t)}{\partial t} = (1-\lambda)\gamma^-(t)\frac{\partial F(\lambda, t)}{\partial \lambda} + \sum_{k=1}^{\infty} \gamma_k^+(t)(\lambda^k - 1)F(\lambda, t). \tag{2.14}$$

Now we make a *convoluted multiple Poisson ansatz* which satisfies the boundary and initial conditions

$$F(\lambda, t) = \exp\left(\sum_{k=1}^{\infty} C(k, t)(\lambda^k - 1)\right) \tag{2.15}$$

where $C(k, t=0)=0$, $k=1, 2, \ldots$. This results in the coupled system of equations

$$\frac{\mathrm{d}C(k, t)}{\mathrm{d}t} = \gamma_k^+(t) - \gamma^-(t)[kC(k, t) - (k+1)C(k+1, t)] \qquad \text{for } k=1, 2, \ldots. \tag{2.16}$$

We may immediately note that if $\gamma_q^+(t)=0$ for all $q \geqslant k$, then $C(q, t)$ must vanish as well for all $q \geqslant k$. So the number of combinants in the solution does not exceed the largest number of bosons which are instantaneously emitted as a group (dynamically correlated). In particular, if the bosons are all emitted singly (dynamically uncorrelated), this model yields an ordinary Poisson multiplicity distribution, although with a mean varying in time according to the equation

$$\frac{\mathrm{d}\bar{n}(t)}{\mathrm{d}t} = \gamma_1^+(t) - \gamma^-(t)\bar{n}(t) \tag{2.17a}$$

and satisfying the initial condition

$$\bar{n}(t=0) = 0. \tag{2.17b}$$

Provided that $\gamma_k^+(t)$ and $\gamma^-(t)$ approach appropriate limits at large times, we may solve for the *equilibrium* combinants

$$C(k, t=+\infty) = \frac{1}{k}\sum_{q=k}^{\infty}\left(\frac{\gamma_q^+(+\infty)}{\gamma^-(+\infty)}\right). \tag{2.18}$$

Or course, if only the instantaneous *single* boson emission rate persists at large times, the equilibrium distribution is an ordinary Poisson with mean

$$\bar{n}=\frac{\gamma_1^+(+\infty)}{\gamma^-(+\infty)}. \tag{2.19}$$

In the more general case that the instantaneous rate for emitting a *group* of k bosons persists at large times, its ratio to the single boson absorption rate at large times is given by a simple function of the equilibrium combinants:

$$D(k)\equiv[kC(k,t=+\infty)-(k+1)C(k+1,t=+\infty)] \tag{2.20a}$$

$$=\frac{\gamma_k^+(+\infty)}{\gamma^-(+\infty)}. \tag{2.20b}$$

If we return to the 'independent Bose–Einstein per mode' class of models, we will note that for these, $D(k)\equiv[kC(k)-(k+1)C(k+1)]$ is just the mean number of modes, in the range of interest, which have occupation number k. So, in both of these models, $D(k)$, a simple function of $C(k)$ and $C(k+1)$, provides an indication of the degree to which the bosons are *correlated into groups of k*. It is tempting to assign $D(k)$ the role of *indicator* for k-fold boson correlations in the general instance (a negative value for $D(k)$ would indicate a k-fold anticorrelation). The validity of such an interpretation of $D(k)$ in a given situation seems to be related, however, to the applicability of classical probability notions to that situation. An especially severe counter example is provided by the 'coherent state' Poisson distribution of bosons produced by a single classical source current—here there can be large 'correlated' populations per mode, as in bosons from a laser, but still with a Poisson distribution, which has all $C(k)$ and $D(k)$ vanishing for $k\geqslant 2$.

We have discussed some models of created boson multiplicity distributions with the aim of illustrating the natural applicability and power of combinants in a variety of theoretical approaches. This, together with the natural relation of combinants to the finite set of multiplicity frequencies which make up experimental data, should make combinant analysis a standard tool for both theoretical and experimental study of most created boson multiplicity distributions.

Since combinants thus have an important role in the study of created boson multiplicity distributions, we present, in the next section, formulae for the traditional, moment based, probability coefficients of the closely related general convoluted multiple Poisson.

3. Moments and related probability coefficients of the convoluted multiple Poisson

The convoluted multiple Poisson is the most general form of a created boson multiplicity distribution, so it is worthwhile, for completeness, to display its moments and related 'ordinary' probability coefficients. The formulae thus developed relate these traditional, moment based, coefficients directly to the combinants.

To obtain the moments of the convoluted multiple Poisson, we note the standard relation between the generating function $F(\lambda)$ and the *moment* generating function

$$f(\alpha)\equiv\sum_{m=0}^{\infty}\frac{\langle n^m\rangle}{m!}\alpha^m \tag{3.1a}$$

$$=F(\mathrm{e}^{\alpha}). \tag{3.1b}$$

Now we obtain $F(e^\alpha)$ for the general convoluted multiple Poisson from equation (1.8*b*) and then proceed to use our standard techniques: insertion of one series into another, use of the multinomial expansion, and re-arrangement into ascending powers of α in order to 'pick off' the $\langle n^m\rangle$:

$$\sum_{m=0}^{\infty}\frac{\langle n^m\rangle}{m!}\alpha^m=\exp\Big(\sum_{k=1}^{\infty}C(k)(e^{k\alpha}-1)\Big) \tag{3.2a}$$

$$=\exp\Big(\sum_{k=1}^{\infty}C(k)\sum_{j=1}^{\infty}\frac{(k\alpha)^j}{j!}\Big) \tag{3.2b}$$

$$=\sum_{q=0}^{\infty}\frac{1}{q!}\Big[\sum_{j=1}^{\infty}\alpha^j\Big(\frac{1}{j!}\sum_{k=1}^{\infty}k^jC(k)\Big)\Big]^q \tag{3.2c}$$

$$=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\Big\{\prod_{j=1}^{\infty}\Big[\frac{(\alpha^j)^{n_j}}{n_j!}\Big(\frac{1}{j!}\sum_{k=1}^{\infty}k^jC(k)\Big)^{n_j}\Big]\Big\} \tag{3.2d}$$

$$=\sum_{m=0}^{\infty}\alpha^m\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\Big\{\prod_{j=1}^{\infty}\Big[\frac{1}{n_j!}\Big(\frac{1}{j!}\sum_{k=1}^{\infty}k^jC(k)\Big)^{n_j}\Big]\Big\}\delta\Big(m,\sum_{r=1}^{\infty}rn_r\Big). \tag{3.2e}$$

Thus, we finally obtain for the mth moment

$$\langle n^m\rangle=m!\sum_{n_1=0}^{m}\cdots\sum_{n_p=0}^{[m/p]}\cdots\sum_{n_m=0}^{1}\Big\{\prod_{j=1}^{m}\Big[\frac{1}{n_j!}\Big(\frac{1}{j!}\sum_{k=1}^{\infty}k^jC(k)\Big)^{n_j}\Big]\Big\}\delta\Big(m,\sum_{r=1}^{m}rn_r\Big),$$
$$m=1,2,\ldots. \tag{3.3}$$

For the factorial moments, it is possible to proceed in much the same fashion. We write down the relation between the factorial moment generating function and the generating function:

$$g(t)\equiv\sum_{m=0}^{\infty}\Big\langle\frac{n!}{(n-m)!}\Big\rangle\frac{t^m}{m!} \tag{3.4a}$$

$$=F(1+t). \tag{3.4b}$$

We now proceed in strict analogy with the steps of equation (3.2). Indeed, most of the steps are sufficiently similar that we leave them out:

$$\sum_{m=0}^{\infty}\Big\langle\frac{n!}{(n-m)!}\Big\rangle\frac{t^m}{m!}$$

$$=\exp\Big(\sum_{k=1}^{\infty}C(k)[(1+t)^k-1)]\Big) \tag{3.5a}$$

$$=\exp\Big[\sum_{k=1}^{\infty}C(k)\sum_{j=1}^{k}\binom{k}{j}t^j\Big] \tag{3.5b}$$

$$=\exp\Big[\sum_{j=1}^{\infty}t^j\Big(\frac{1}{j!}\sum_{k=j}^{\infty}\frac{k!}{(k-j)!}C(k)\Big)\Big] \tag{3.5c}$$

$$=\sum_{m=0}^{\infty}t^m\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\Big\{\prod_{j=1}^{\infty}\Big[\frac{1}{n_j!}\Big(\frac{1}{j!}\sum_{k=j}^{\infty}\frac{k!}{(k-j)!}C(k)\Big)^{n_j}\Big]\Big\}\delta\Big(m,\sum_{r=1}^{\infty}rn_r\Big). \tag{3.5d}$$

The result for the mth factorial moment is

$$\left\langle\frac{n!}{(n-m)!}\right\rangle=m!\sum_{n_1=0}^{m}\cdots\sum_{n_p=0}^{[m/p]}\cdots\sum_{n_m=0}^{1}\left\{\prod_{j=1}^{m}\left[\frac{1}{n_j!}\left(\frac{1}{j!}\sum_{k=j}^{\infty}\frac{k!}{(k-j)!}C(k)\right)^{n_j}\right]\right\}\delta\left(m,\sum_{r=1}^{m}rn_r\right),$$
$$m=1,2,\ldots. \tag{3.6}$$

We turn now to the cumulants. The relation between the generating function and the cumulant generating function is (Burington and May 1953)

$$\omega(\alpha)\equiv\sum_{j=1}^{\infty}\frac{\kappa_j}{j!}\alpha^j \tag{3.7a}$$
$$=\ln(F(\mathrm{e}^{\alpha})). \tag{3.7b}$$

We proceed as in the first few steps of equation (3.2)

$$\sum_{j=1}^{\infty}\frac{\kappa_j}{j!}\alpha^j=\sum_{k=1}^{\infty}C(k)(\mathrm{e}^{k\alpha}-1) \tag{3.8a}$$
$$=\sum_{j=1}^{\infty}\frac{\alpha^j}{j!}\left(\sum_{k=1}^{\infty}k^jC(k)\right). \tag{3.8b}$$

We now have a simple and suggestive result for the jth cumulant

$$\kappa_j=\sum_{k=1}^{\infty}k^jC(k),\qquad j=1,2,\ldots. \tag{3.9}$$

We work out the factorial cumulants in an analogous fashion. We write down the relation between the generating function and the factorial cumulant generating function (Burington and May 1953):

$$\phi(t)\equiv\sum_{j=1}^{\infty}\frac{\chi_j}{j!}t^j \tag{3.10a}$$
$$=\ln(F(1+t)). \tag{3.10b}$$

We proceed as in the first few steps of equation (3.5)

$$\sum_{j=1}^{\infty}\frac{\chi_j}{j!}t^j=\sum_{k=1}^{\infty}C(k)((1+t)^k-1) \tag{3.11a}$$
$$=\sum_{j=1}^{\infty}\frac{t^j}{j!}\left(\sum_{k=j}^{\infty}\frac{k!}{(k-j)!}C(k)\right). \tag{3.11b}$$

Again, the result for the jth factorial cumulant is elegant

$$\chi_j=\sum_{k=j}^{\infty}\frac{k!}{(k-j)!}C(k),\qquad j=1,2,\ldots. \tag{3.12}$$

The cumulants (both ordinary and factorial) have exactly the same formal relation to the combinants as the moments (ordinary and factorial, respectively) have to the probabilities. (Of course, the combinants, some of which may be negative, are *not* probabilities.) The moments do not have a *simple* representation in terms of the combinants, but this is just a consequence of the fact that the moments bear a complicated relationship to the cumulants.

Among the 'ordinary' probability coefficients, the cumulants are the most closely related to the combinants, but it seems fair to assert that between the two, the combinants are the more elementary.

Appendix

Many of the finite sums associated with combinants, including the fundamental relation of equation (1.15), can be readily programmed on a computer only with the aid of an algorithm which sequentially generates the full set of non-negative integer k-tuples $(n_1, n_2, \ldots, n_k)$ which satisfy

$$\sum_{r=1}^{k} rn_r = k. \tag{A.1}$$

Each such k-tuple generally corresponds to a single term of the sum, and it may be readily employed to generate this term as a product of factors. After accumulating the term, the process is to be repeated with the 'next' k-tuple. We shall present the steps of a scheme which sequentially generates the k-tuples in a fixed order (calendar ordering), beginning with $(0, 0, \ldots, 0, 1)$, and terminating with $(k, 0, 0, \ldots, 0)$.

We assume that we are *given* the integer variable k having some positive integer value, and we work with the integer array $(n_1, n_2, \ldots, n_k, \ldots)$ of dimension greater than or equal to k, the integer 'pointer' variable r, and the integer 'remainder' variable j. One underlying theme of the approach is that, at every stage, all array elements to the 'left' of the 'pointer' are zero. Another, is that

$$k - \sum_{r=1}^{k} rn_r = j \geqslant 0, \tag{A.2}$$

and the scheme proceeds to reduce j to zero through appropriate changes of array elements indicated by the 'pointer', which itself is varied in a systematic, incremental way. The detailed steps follow.

(A.3)

1. Set n_r to 0 for $r = 1, 2, \ldots, k$.
2. Set r to k.
3. Set n_r to 1.
4. Set j to 0.
5. Go to step 18.
6. If $r \geqslant 2$, go to step 12.
7. Set j to n_r.
8. Set n_r to 0.
9. Set r to $r+1$.
10. If $r > k$, go to step 20.
11. If $n_r = 0$, go to step 9.
12. Set n_r to $n_r - 1$.
13. Set j to $j + r$.
14. Set r to $r - 1$.
15. Set n_r to $[j/r]$.
16. Set j to $j \bmod r$.
17. If $j > 0$, go to step 14.

18. Make use of the current k-tuple $(n_1, n_2, \ldots, n_k)$, but leave it unaltered. Also, do not change the values of k, r, or j.
19. Go to step 6.
20. Task completed. All the k-tuples have been generated, and at this point, $n_r = 0$ for $r = 1, 2, \ldots, k$ and $r = k + 1$.

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